Stokes flow of a conducting fluid past an axially symmetric body in the presence of a uniform magnetic field

By I-DEE CHANG
Guggenheim Aeronautical Laboratory, California Institute of Technology

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Low Reynolds number flow of an incompressible fluid past an axially symmetric body in the presence of a uniform magnetic field is studied using a perturbation method. It is found that for small Hartmann number M an approximate drag formula is given by

$$D' = D'_0\left(1+\frac{D'_0}{16\pi\rho\nu a U}M\right)+O(M^2),$$

where D'_0 is the Stokes drag for flow with no magnetic effect.

1. Introduction

In 1957, Chester studied low Reynolds number flow of an incompressible conducting fluid past a sphere in a magnetic field which is uniform at infinity. He showed that when the magnetic Reynolds number R_m is small the magnetic field is essentially independent of the fluid motion. For the case where the solid and the fluid have nearly the same permeability, a uniform magnetic field will result, i.e. $\mathbf{H}' = H_0\mathbf{i}$ = magnetic field at infinity. It follows further from the symmetry that there is no electric field, since for all such flows the electric currents form closed circuits. The governing equations and the boundary conditions for the problem become

$$\operatorname{div}\mathbf{v} = 0, \tag{1a}$$

$$-\nabla p+\nabla^2\mathbf{v}-M^2[\mathbf{v}-(\mathbf{v}.\mathbf{i})\,\mathbf{i}] = 0, \tag{1b}$$

$$\mathbf{v}\to\mathbf{i} \quad \text{as} \quad r\to\infty \quad (r^2 = x^2+y^2+z^2), \tag{2a}$$

$$\mathbf{v} = 0 \text{ at the body.} \tag{2b}$$

In the above, all quantities are non-dimensional;

U = free-stream velocity;

a = characteristic length of body;

$\mathbf{v} = \dfrac{\mathbf{v}'}{U},\ p = \dfrac{a(p'-p'_\infty)}{\rho\nu U},\quad x = \dfrac{x'}{a}$, etc.;

$Re = \dfrac{Ua}{\nu}$ = Reynolds number;

$R_m = Ua\mu\sigma$ = magnetic Reynolds number;

$M = \mu H_0 a\left(\dfrac{\sigma}{\rho\nu}\right)^{\frac{1}{2}}$ = Hartmann number;

$\mathbf{i}$ = unit vector in the x-direction.

Other symbols have the usual meanings in electro- and hydrodynamics. Primed quantities are in physical units (cf. Chester 1957).

In this note we consider the more general problem of flow past an axially symmetric body. Assuming that the Hartmann number M is small, we shall show that the drag on the body is given by the formula

$$D \equiv \frac{D'}{\rho\nu Ua} = D_0\left(1+\frac{D_0}{16\pi}M\right)+O(M^2). \tag{3}$$

$D_0 = D_0'/\rho\nu Ua$ is the (non-dimensional) drag on the body in Stokes flow without magnetic effect and is known by existing formulas, such as those given by Payne & Pell (1960). To the order of approximation $O(M)$, D_0 also appears as a parameter characterizing the body shape.

2. The perturbation method

Following the usual procedure, one considers the following expansions of the exact solutions of (1):

$$\mathbf{v} = \mathbf{h}^{(0)}(x,y,z)+M\mathbf{h}^{(1)}(x,y,z)+M^2\mathbf{h}^{(2)}(x,y,z)+\ldots, \tag{4a}$$

$$p = p^{(0)}(x,y,z)+Mp^{(1)}(x,y,z)+M^2p^{(2)}(x,y,z)+\ldots. \tag{4b}$$

By insertion of the above expansions into equation (1) one obtains the following equations:

$$O(1); \quad \operatorname{div}\mathbf{h}^{(0)} = 0, \tag{5a}$$

$$-\nabla p^{(0)}+\nabla^2\mathbf{h}^{(0)} = 0; \tag{5b}$$

$$O(M); \quad \operatorname{div}\mathbf{h}^{(1)} = 0, \tag{6a}$$

$$-\nabla p^{(1)}+\nabla^2\mathbf{h}^{(1)} = 0; \tag{6b}$$

$$O(M^2); \quad \operatorname{div}\mathbf{h}^{(2)} = 0, \tag{7a}$$

$$-\nabla p^{(2)}+\nabla^2\mathbf{h}^{(2)} = \mathbf{h}^{(0)}-(\mathbf{h}^{(0)}.\mathbf{i})\,\mathbf{i}, \tag{7b}$$

etc. Equations (5) and (6) are identical to the Stokes equations. This indicates that at a finite distance from the body, i.e., $r = O(1)$, in the limit $M \to 0$, the flow field is essentially governed by Stokes's equations. However, it can be shown that at great distances from the body (i.e. $r = O(M^{-\alpha})$, $0 < \alpha \leqslant 1$) the above expansions given by (4) are not uniform approximations of the exact solutions; by not being uniform approximations we mean that the difference between the expansions and the exact solutions for a given value of M becomes arbitrarily large as $r \to \infty$. A uniformly valid approximation of the exact solution in the neighbourhood of $r = \infty$ is given by a second type of expansion using different variables (outer expansions). The expansions defined by (4), which are uniformly valid in the neighbourhood of the body, are referred to as the inner expansions.

The outer expansions are defined by

$$\mathbf{v} = \mathbf{i}+M\mathbf{g}^{(1)}(\tilde{x},\tilde{y},\tilde{z})+M^2\mathbf{g}^{(2)}(\tilde{x},\tilde{y},\tilde{z})+\ldots, \tag{8a}$$

$$p = M^2\tilde{p}^{(1)}(\tilde{x},\tilde{y},\tilde{z})+M^3\tilde{p}^{(2)}(\tilde{x},\tilde{y},\tilde{z})+\ldots, \tag{8b}$$

and the independent variables are now (outer variables)

$$\tilde{x} = Mx, \quad \tilde{y} = My, \quad \tilde{z} = Mz. \tag{9}$$

The above expansions given by (8) may be formally obtained from the exact solutions by the limit process $M \to 0$ with fixed $(\tilde{x},\tilde{y},\tilde{z})$ and are expected to be uniformly valid for all $r = O(M^{-1})$. Insertion of the expansions into (1) leads to the equations

$$\widetilde{\operatorname{div}}\, \mathbf{g}^{(i)} = 0 \quad (i = 1, 2, \ldots), \tag{10a}$$

$$-\tilde{\nabla} p + \tilde{\nabla}^2 \mathbf{g}^{(i)} - [\mathbf{g}^{(i)} - (\mathbf{g}^{(i)}.i)\, i] = 0. \tag{10b}$$

The equations that determine each individual term of the outer expansions (8) are thus identical to the exact equations (1); however, we shall see below that the boundary conditions for these equations are simpler than those for equations (1).

There remains the problem of determining the proper boundary conditions for the individual terms of the inner and the outer expansions. Since the inner expansions are not valid at infinity, boundary condition (2a) does not in general hold for the inner expansions. For a similar reason, boundary condition (2b) in general does not hold for the outer expansions. These boundary conditions are replaced by the matching conditions, which have the requirements that the inner and the outer expansions should agree term for term for some intermediate orders of r, namely $r = O(M^{-\alpha})$, where $0 < \alpha < 1$, for which the inner and the outer expansions are both valid. The use of the matching conditions will become clear from the way in which the solutions are determined (cf. Lagerstrom & Cole 1955; Proudman & Pearson 1957; Lagerstrom).

3. The first-order inner solutions

The first term in the outer expansion (8a) is the free-stream velocity $\mathbf{v} = \mathbf{i}$. This term may be understood by the following reasoning. Assume the solid is a sphere of radius a. Then the boundary of the solid is given in outer variables by $\tilde{r} = M$ and in the limit $M \to 0$ the body shrinks to a point. A point cannot cause a finite disturbance in the fluid, hence the value of $\mathbf{v}$ will tend to the free-stream velocity $\mathbf{i}$.

Let us consider the first-order inner solution $\mathbf{h}^{(0)}$ and $p^{(0)}$. For the inner solutions, the no-slip boundary conditions are valid:

$$\mathbf{h}^{(i)} = 0 \quad \text{at the body.} \tag{11}$$

By the matching conditions, $\mathbf{h}^{(0)}$ must agree for large values of r with the leading term of the outer expansion, and hence

$$\mathbf{h}^{(0)} \to \mathbf{i} \quad \text{as} \quad r \to \infty. \tag{12}$$

Equations (5), (11) and (12) show that the solutions of $\mathbf{h}^{(0)}$ and $p^{(0)}$ are simply the solutions of the non-magnetic Stokes problems.

For a sphere, such solutions are

$$\mathbf{h}^{(0)} = \mathbf{i} - \tfrac{3}{2}\mathbf{A}_1 + \tfrac{1}{4}\mathbf{A}_2,$$

where

$$\mathbf{A}_1 = \frac{\mathbf{i}}{r} - \nabla\frac{x}{2r}, \quad \mathbf{A}_2 = \nabla\frac{\partial}{\partial x}\frac{1}{r},$$

and

$$p = -\frac{3x}{2r^3}.$$

The drag on the sphere, in physical units, is

$$D_0' = 6\pi\rho\nu U a.$$

Explicit solutions are also available for bodies of many other shapes (Oseen 1927; Payne & Pell 1960). It can be shown in general that for large values of r the asymptotic expressions of the Stokes solutions for axially symmetric bodies of any shape are given by

$$\mathbf{h}^{(0)} = \mathbf{i} - \frac{D_0}{4\pi}\left(\frac{\mathbf{i}}{r} - \nabla\frac{x}{2r}\right) + O\left(\frac{1}{r^2}\right), \tag{13a}$$

$$p^{(0)} = -\frac{D_0 x}{4\pi r^3} + O\left(\frac{1}{r^3}\right) \tag{13b}$$

(Odqvist 1930; Payne & Pell 1960). If one defines $r_\alpha = M^\alpha r$, where $0 < \alpha < 1$, and rewrites (13) in this variable, the following results are obtained:

$$\mathbf{h}^{(0)}(x_\alpha, r_\alpha; M) = \mathbf{i} - \frac{D_0}{4\pi}\left(\frac{\mathbf{i}}{r_\alpha} - \nabla_\alpha \frac{x_\alpha}{2r_\alpha}\right) M^\alpha + O(M^{2\alpha}), \tag{14a}$$

$$p^{(0)}(x_\alpha, r_\alpha; M) = -\frac{D_0 x_\alpha}{4\pi r_\alpha^3} M^{2\alpha} + O(M^{3\alpha}). \tag{14b}$$

Equations (14) imply that, when r is of an intermediate order $O(M^{-\alpha})$, $0 < \alpha < 1$, in comparison with that used in the inner ($\alpha = 0$) and outer ($\alpha = 1$) limits, the leading terms of the expansion of $\mathbf{h}^{(0)}$ and $p^{(0)}$ are the terms displayed in (13). The first-order solutions of the outer expansions, $\mathbf{g}^{(1)}$ and $\tilde{p}^{(1)}$ must therefore agree with the inner solutions in these terms.

4. The first-order outer solutions

The solutions of $\mathbf{g}^{(1)}$ and $\tilde{p}^{(1)}$ are determined from equations (10) with the boundary condition $\mathbf{g}^{(1)} \to 0$ as $\tilde{r} \to \infty$ and the matching condition stated at the end of § 3. Such solutions are given by

$$\mathbf{g}^{(1)} = D_0 \mathbf{G}, \quad \tilde{p}^{(1)} = D_0 P, \tag{15a, b}$$

where

$$\mathbf{G} = -\frac{1}{4\pi}\left[\frac{\mathbf{i}}{\tilde{r}} e^{-\frac{1}{2}\tilde{r}} \cosh\frac{\tilde{x}}{2} - \tilde{\nabla}\frac{1}{\tilde{r}} e^{-\frac{1}{2}\tilde{r}} \sinh\frac{\tilde{x}}{2}\right], \tag{16a}$$

$$P = \frac{1}{4\pi}\left[\frac{\partial}{\partial\tilde{x}}\frac{1}{\tilde{r}} e^{-\frac{1}{2}\tilde{r}} \cosh\frac{\tilde{x}}{2} - \frac{1}{\tilde{r}} e^{-\frac{1}{2}\tilde{r}} \sinh\frac{\tilde{x}}{2}\right]. \tag{16b}$$

When (16) are rewritten in the variable r_α and expanded in powers of M, there are obtained the results

$$MG = -\frac{1}{4\pi}\left(\frac{\mathbf{i}}{r_\alpha} - \nabla_\alpha \frac{x_\alpha}{2r_\alpha}\right) M^\alpha + \frac{\mathbf{i}}{16\pi} M + O(M^{2-\alpha}), \tag{17a}$$

$$M^2 P = -\frac{x_\alpha}{4\pi r_\alpha^3} M^{2\alpha} + O(M^2). \tag{17b}$$

By comparing these results with (14), one sees that the inner and the outer expansions are matched to the order $O(M^\alpha)$.

5. The drag on the body

The leading term in the outer expansion of $\mathbf{v}$ which is not matched by the inner expansion is now of order $O(M)$. From (17 a) it follows that the second-order inner solution $\mathbf{h}^{(1)}$, which satisfies the Stokes equations (6 b), should satisfy the boundary conditions $\mathbf{h}^{(1)} = 0$ at the body and the condition $\mathbf{h}^{(1)} \to D_0 \mathbf{i}/16\pi$ as $r \to \infty$. Such a solution is easily seen to be

$$\mathbf{h}^{(1)} = \frac{D_0}{16\pi}\mathbf{h}^{(0)}. \tag{18a}$$

The corresponding pressure is

$$p^{(1)} = \frac{D_0}{16\pi} p^{(0)}. \tag{18b}$$

To the order $O(M)$ the inner solution of the velocity field is then

$$\mathbf{v} = \left(1 + \frac{D_0}{16\pi} M\right) \mathbf{h}^{(0)} + O(M^2), \tag{19}$$

Body	Drag (non-magnetic)	Drag (magnetic)
Hemispherical cup (radius a)	$17{\cdot}525\rho\nu aU$	$17{\cdot}525\rho\nu aU\left(1 + \frac{1{\cdot}095}{\pi} M\right)$
Flat disc (radius a)	$16\rho\nu aU$	$16\rho\nu aU\left(1 + \frac{1}{\pi} M\right)$
Sphere (radius a)	$6\pi\rho\nu aU$	$6\pi\rho\nu aU(1 + \frac{3}{8}M)$
Prolate spheroid*	$8\pi\delta\rho\nu aU$	$8\pi\delta\rho\nu aU(1 + \frac{1}{2}\delta M)$
Oblate spheroid*	$8\pi\beta\rho\nu aU$	$8\pi\beta\rho\nu aU(1 + \frac{1}{2}\beta M)$

* δ and β are constants related to the geometry of the spheroid (Payne & Pell 1960).

TABLE 1

which shows that the effect of the magnetic field on the inner Stokes flow is an apparent increase of the free-stream velocity in the ratio $1 : \left(1 + \frac{D_0}{16\pi} M\right)$. There is a corresponding amount of increase in the drag on the body, as can be easily verified by simple argument. The drag on the body is then given by formula (3).

In Table 1 we list a few cases for which the non-magnetic Stokes drag formulas are given by Payne & Pell (1960).

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